Math 1B Chapter 8.1-8.4 Test Solutions – Fall ’08

Test each series for convergence. Try to use a different test for each series.

(a) $n$th term test *9*10
(b) geometric series test *1*2*4
(c) $p$-series test *9
(d) telescoping series *3
(e) integral test *7
(f) direct comparison *1*9
(g) limit comparison test *6
(h) root test *4
(i) ratio test *2*4*8
(j) alternating series test *5

1. $\sum_{n=0}^{\infty} \frac{(0.999)^n}{n!}$ converges by direct comparison with the geometric series

$$\sum_{n=0}^{\infty} (0.999)^n = \frac{1}{1-0.999} = 1000$$

2. $\sum_{n=0}^{\infty} \frac{(1.001)^n}{n!}$ converges by the ratio test: $\lim_{n\to\infty} \frac{(1.001)^{n+1}}{(n+1)!} \frac{n!}{(1.001)^n} = \lim_{n\to\infty} \frac{1.001}{n+1} = 0 < 1$

3. $\sum_{n=1}^{\infty} \Delta x\left(\ln(n+1) - \ln n\right)$ diverges as a telescoping series:

$$\sum_{n=1}^{N} \Delta x\left(\ln(n+1) - \ln n\right) = \Delta x \ln(N+1) \to \infty \text{ as } N \to \infty$$

4. $\sum_{k=0}^{\infty} \frac{3^k}{4^{k+1}}$ converges as a geometric series: $\sum_{k=0}^{\infty} \frac{3^k}{4^{k+1}} = \frac{1}{4}\sum_{k=0}^{\infty} \frac{3^k}{4^k} = \frac{1}{4}\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k = \frac{1}{4(1-3/4)} = 1$

5. $\sum_{j=1}^{\infty} \left(-\frac{1}{5}\right)^j - \left(-\frac{1}{6}\right)^j = \sum_{j=1}^{\infty} (-1)^j \left(\frac{6^j - 5^j}{30^j}\right)$ converges by the alternating series test.

6. $\sum_{n=1}^{\infty} \frac{n^3+1}{n^4+n^2-2}$ diverges by limit comparison with the harmonic series:

$$\lim_{n\to\infty} \frac{n^3+1}{n^4+n^2-2} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n^4+n}{n^4+n^2-2} = 1$$

7. $\sum_{k=100^{100}}^{\infty} \frac{0.1}{k}$ diverges as a $p$-series with $p = 1$. Of course, you can also use the integral test.

8. $\sum_{x=1}^{\infty} \frac{e^x}{x!}$ converges by the ratio test: $\lim_{x\to\infty} \frac{e^{x+1}}{(x+1)!} \frac{x!}{e^x} = \lim_{x\to\infty} \frac{e}{x+1} = 0 < 1$ You can also use the root test here: $\lim_{n\to\infty} \sqrt[n]{\frac{e^n}{n!}} = \frac{e}{\lim_{n\to\infty} \sqrt[n]{n!}} = 0$. To see this, use $\ln(n!) \approx \int_0^n \ln t \, dt$ for large $n$ and consider

$$\ln\left(\lim_{n\to\infty} \sqrt[n]{n!}\right) = \lim_{n\to\infty} \ln\left(\sqrt[n]{n!}\right) = \lim_{n\to\infty} \frac{\ln(n!)}{n} = \lim_{n\to\infty} \frac{\int_0^n \ln t \, dt}{n} = \lim_{n\to\infty} \ln n = \infty$$

9. $\sum_{z=2}^{\infty} \frac{z}{\ln z}$ diverges by direct comparison with the harmonic series, since $\frac{z}{\ln z} > \frac{1}{z}$ for $z > 2$

10. $\sum_{z=2}^{\infty} \frac{x^x}{x!}$ diverges by the $n$th term test, since $\frac{x^x}{x!} > 1$ if $x > 1$.